Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(00)00056-3
J. Appl. Maths Mechs, Vol. 64, No. 2, pp. 337–339, 2000

0021–8928/00/$—see front matter

NON-LINEAR INSTABILITY OF STEADY FLOWS GENERATED BY A VORTEX FILAMENT IN STRATIFIED GAS†

Kh. V. OGANESYAN and A. M. TER-KRIKOROV

Dolgoprudnyi

(*Received* 5 *July* 1999)

It is shown that the steady solution of the exact non-linear equation of motion of an ideal, uniformly stratified gas generated by a vortex filament is unstable and the nature of this slowly developing instability is investigated. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

The equation

$$\frac{\partial}{\partial u}\left(\frac{\partial^2 W}{\partial \tau^2}+W+u\right)+\frac{\partial W}{\partial u}\frac{\partial^2 W}{\partial \tau^2}=0 \tag{1.1}$$

$$w=\frac{g}{N^2}W,\quad Nt=\tau,\quad u=-\frac{N^2}{g^2}\int\limits_{+\infty}^{r}\frac{v_\theta^2(x)}{x}dx \tag{1.2}$$

which describes internal waves in an ideal gas filling a three-dimensional space has been derived [1] for the case when the motion possesses axial symmetry, the gas is exponentially stratified and the changes in the speed of sound in the fluid particles are small compared with the speed of sound. This enables one to take the equation for the conservation of mass in the same form as for an incompressible fluid, which is equivalent to neglecting acoustic oscillations compared with buoyancy oscillations. In (1.2), the variable w specifies the deviation along a vertical of the fluid particles from the equilibrium position, g is the acceleration due to gravity, N is the Brunt–Väisälä frequency for an ideal gas, t is the time, r is the distance of a particle from the axis of symmetry and the arbitrary function $v_\theta(r)$ specifies the peripheral velocity distribution. In the case of a vortex filament, $v_\theta(r) = \Gamma/(2/\pi r)$, where Γ is the circulation along a simple contour encompassing the vortex filament, and by virtue of (1.2), $u = N^2\Gamma/(8\pi^2 r^2)$.

Equation (1.1) has a steady solution $W = -u$. It has been shown [1] that the steady solution is linearly unstable. Below, an attempt is made to investigate the stability of the steady solution in a non-linear formulation.

We specify the arbitrary initial perturbations

$$W(\tau,0)=\sum_{k=1}^{\infty}\alpha_k u^k,\quad \frac{\partial W(\tau,0)}{\partial\tau}=\sum_{k=1}^{\infty}\beta_k u^k \tag{1.3}$$

By putting $W = -u + e^u\omega$ in Eqs (1.1) and (1.2), we reduce the initial problem to the Cauchy problem

$$\frac{\partial}{\partial u}(L\omega)+\omega+e^u\left(\frac{\partial\omega}{\partial u}+\omega\right)(L\omega-\omega)=0,\quad L\omega=\frac{\partial^2\omega}{\partial\tau^2}+\omega$$
$$\omega(u,0)=\sum_{k=1}^{\infty}\alpha_k u^k,\quad \frac{\partial\omega(u,0)}{\partial\tau}=\sum_{k=1}^{\infty}(\beta_k-\alpha_k)u^k \tag{1.4}$$

2. SOLUTION OF THE NON-LINEAR PROBLEM

The structure of the previously obtained [1] exact solution of the linearized Cauchy problem (1.4) suggests the idea of finding the solution of the non-linear Cauchy problem (1.4) in the form of the series

$$\omega=\sum_{n=0}^{\infty}u^n\omega_n(\tau,z),\quad z=\sqrt{2u\tau},\quad \omega_0=0 \tag{2.1}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 2, pp. 349–351, 2000.

Substituting expansion (2.1) into Eqs (1.4) and equating the coefficients of like powers of u, we obtain an infinite recurrent system of equations

$$\frac{\partial}{\partial z}(z^{2k+2}(L\omega_{k+1}+P_{k+1}(\omega_k,\omega_{k-1})))+2z^{2k+1}\omega_k+2z^{2k+1}S_k(\omega_1,\ldots,\omega_k)=0 \tag{2.2}$$

where

$$P_k(\omega_{k-1},\omega_{k-2})=\frac{2}{z}\frac{\partial^2\omega_{k-1}}{\partial\tau\partial z}+\frac{1}{z^2}\frac{\partial^2\omega_{k-2}}{\partial z^2}-\frac{1}{z^3}\frac{\partial\omega_{k-2}}{\partial z}$$

$$S_n=\sum_{k=0}^{n}R_{n-k}(L\omega_k-\omega_k P_k)$$

$$Q_k(\omega_k,\omega_{k+1})=\frac{1}{2z^{2k+1}}\frac{\partial}{\partial z}(z^{2k+2}\omega_{k+1}(\tau,z))+\omega_k$$

$$R_k(\omega_1,\ldots,\omega_{k+1})=\sum_{m=0}^{k}\frac{(-1)^{k-m}}{(k-m)!}Q_m(\omega_m,\omega_{m+1})u^m$$

By virtue of (1.4), the initial conditions are

$$\omega_k(0)=\alpha_k,\quad \partial\omega_k(0)/\partial t=\beta_k-\alpha_k \tag{2.3}$$

The first two equations of (2.2) have the form

$$\frac{\partial}{\partial z}(z^2L\omega_1)=0 \tag{2.4}$$

$$\frac{\partial}{\partial z}\left(z^4L\omega_2+2z^3\frac{\partial^2\omega_1}{\partial\tau\partial z}\right)+2z^3\omega_1+z^2\frac{\partial}{\partial z}(z^2\omega_1)(L\omega_1-\omega_1)=0 \tag{2.5}$$

Since $z^2 = u\tau$, $u\omega_1 = 2c/\tau$, we obtain a bounded solution of Eq. (2.4) in the form

$$\omega_1=\frac{a'(z)}{z^3}\cos\tau+\frac{b'(z)}{z^3}\sin\tau \tag{2.6}$$

where $a(z)$, $b(z)$ are arbitrary functions.

In the same way as for Eq. (2.4), Eq. (2.5) reduces to the form

$$z^4L\omega_2+2z^3\frac{\partial^2\omega_1}{\partial\tau\partial z}+2a(z)\cos\tau+2b(z)\sin\tau-\frac{1}{2}(z^2\omega_1^2)=0 \tag{2.7}$$

In order that secular terms should not occur, the coefficients of the cosine and sine must vanish and, hence, the unknown functions must satisfy the system of equations

$$a''-\frac{3a'}{z}-b=0,\quad b''-\frac{3b'}{z}+a=0$$

The function $F(z) = (b + ai)z^{-2}$ satisfies the Bessel equation

$$z^2F''+zF'-(iz^2+4)=0$$

and, hence, the functions $a(z)$, $b(z)$ are expressed in terms of Kelvin functions [2]

$$a(z)=Az^2\,\mathrm{bei}_2\,z+Bz^2\,\mathrm{ber}_2\,z,\quad b(z)=Az^2\,\mathrm{ber}_2\,z-Bz^2\,\mathrm{bei}_2\,z$$
$$\omega_1(t,z)=a'(z)z^{-3}+b'(z)z^{-3}$$

By using the expansion of the Kelvin functions in power series, we obtain from (2.6), for small values of $z^2 = ut$, after using the initial conditions,

$$\omega_1 = \left(-\alpha_1\left(-1+\frac{z^4}{384}+\ldots\right)+(\beta_1-\alpha_1)\left(\frac{z^2}{16}-\frac{z^6}{18432}+\ldots\right)\right)\cos\tau +$$
$$+\left(-\alpha_1\left(\frac{z^2}{16}-\frac{z^6}{18432}+\ldots\right)-(\beta_1-\alpha_1)\left(-1+\frac{z^4}{384}+\ldots\right)\right)\sin\tau$$

Using the asymptotic expressions for Kelvin functions in the case of large values of the argument, we find

$$\omega_1(t,z) \approx \frac{e^{z/\sqrt{2}}}{2z\sqrt{\pi z}}\left(A\cos\tau\sin\left(\frac{z}{2}+\frac{9\pi}{8}\right)+B\sin\tau\cos\left(\frac{z}{2}+\frac{9\pi}{8}\right)\right)$$
$$z=\sqrt{2uNt}, \quad ut\to+\infty$$

It follows from this formula that, when $ut \to +\infty$, the solution tends to infinity. The oscillations therefore become unbounded for long times. However, the oscillations will be bounded in the case of bounded values of $z^2 = ut$.

We will now find the function $\omega_2(z)$. From Eq. (2.7), we obtain

$$\omega_2(z) = \frac{a'^2+b'^2}{z^5}+\frac{a'^2-b'^2}{3z^5}\cos 2\tau+\frac{2a'b'}{3z^5}\sin 2\tau + a_2(z)\cos\tau + b_2(z)\sin\tau$$

The unknown functions $a_2(z)$, $b_2(z)$ are determined from the conditions for the equations of the third approximation to be solvable. Hence, the equilibrium state is also unstable in the non-linear formulation, but this instability develops slowly.

Since the functions ω_n increase exponentially when $z \to +\infty$, series (2.1) will probably converge when the quantity u, defined by equality (1.2), is sufficiently small and the magnitude of z, defined by equality (2.1), is of the order of unity.

It is likely that the instability of the steady solution is due to the fact that the vortex filament is considered in an unbounded space. It would be interesting to investigate the stability in the case of a medium which is bounded by one or two rigid horizontal planes.

REFERENCES

1. OGANESYAN, KH V. and TER-KRIKOROV, A. M., The instability of steady flows generated by a vortex filament in a stratified gas. *Prikl. Mat. Mekh.*, 1999; **63**, 3, 467–469.
2. ABRAMO WITZ, M. and STEGUN, I. Eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* Jov. Print Off., Washington, 1964.

Translated by E.L.S